

Hints of (trans-Planckian) asymptotic freedom in semiclassical cosmology

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Abstract. We employ the semiclassical approximation to the Wheeler-DeWitt equation in the spatially flat de Sitter Universe to investigate the dynamics of a minimally coupled scalar field near the Planck scale. We find that, contrary to naïve intuition, the effects of quantum gravitational fluctuations become negligible and the scalar field states asymptotically approach plane-waves at very early times. These states can then be used as initial conditions for the quantum states of matter to show that each mode essentially originated in the minimum energy vacuum. Although the full quantum dynamics cannot be solved exactly for the case at hand, our results can be considered as supporting the general idea of asymptotic safety in quantum gravity.

1. Introduction

A proper treatment of any matter-gravity system would require the quantisation of all degrees of freedom of matter and gravity. Attempts in this direction include String Theory [1] and Loop Quantum Gravity [2]. While awaiting for a universally accepted theory of quantum gravity, many useful results have been hitherto obtained by quantising the matter fields on a classical background [3]. The limit of this approach can be naïvely identified with the requirement that only energies (lengths) below (above) the Planck scale should be (explicitly) involved in the description of physical processes. There are however notorious exceptions to this rule, namely the occurrence of trans-Planckian frequencies in the Hawking effect [4] and in inflationary models [5], the latter being the case we shall consider hereafter.

According to the above remark, in a Friedman-Robertson-Walker Universe, physical wavelengths of matter fields should be allowed only if $a/k \gtrsim \ell_P$ (k being the wavenumber) or, equivalently, the cosmic scale factor a may be treated as a classical background quantity only when it is larger than multiples of the Planck length $k \ell_P$ for the modes k of interest. Such a condition is easily violated during the inflationary evolution, since modes that are detected in the CMB spectrum today were trans-Planckian at the beginning of inflation [5]. Nonetheless, the results obtained from quantum field theory are in very good agreement with observation.

Many papers can be found in the literature which try to explain why such an approximate method works so well by showing that possible corrections to the CMB spectrum due to the unknown trans-Planckian physics should be small [6, 7, 8]. For example, in Ref. [6], it was suggested that the trans-Planckian phase of the evolution

results into suitable initial conditions for the quantum matter states at the time when the corresponding field modes become sub-Planckian and can afterwards be analysed by the usual means. Several proposals for the initial conditions were also put forward based on the principle of minimum uncertainty [6] or otherwise [8]. There are two main critiques to this viewpoint: since different modes become sub-Planckian at different times, one must assume that the background metric is already classical *before* any matter modes become sub-Planckian (tantamount to requiring that quantum gravitational fluctuations be small); moreover, any choice of initial conditions appears arbitrary without a more explicit knowledge of the trans-Planckian theory. Although several approaches support the possibility that the Universe was indeed born in the semiclassical regime (with a non-zero initial scale factor [9, 10, 11]), the issue of initial conditions for the matter states remains open.

We shall here investigate the dynamics of matter fields in cosmology near the limit of validity of the semiclassical approximation, that is for $a \sim k \ell_P$ when the proper frequency $\omega \sim k/a$ of matter modes approaches the Planck scale. In order to perform our analysis, we shall employ the semiclassical approximation to the quantum Wheeler-DeWitt equation [12] in minisuperspace as proposed, for example, in Ref. [13] and later developed in Refs. [14, 15, 16] (for a recent review of the Wheeler-DeWitt equation in cosmology, see Ref. [17]). In particular, our starting point will be a modified Schrödinger equation for the quantum state of a mode of a minimally coupled scalar field in the de Sitter Universe as obtained within this approach in Ref. [16]. This equation contains terms of order ℓ_P^2 which were treated as a perturbation at large times (*i.e.*, for large scale factor $a/k \gg \ell_P$), and the corresponding corrections to the dispersion relation and CMB power spectrum estimated. We shall show in the next Section that, in the opposite regime of very early times, quantum gravitational fluctuations become negligible and the scalar field is asymptotically free at higher and higher energies. We shall also show that the initial state of each mode is very close to the usual vacuum and the coupling to gravitational perturbations always remains small. Of course, Planckian physics is expected to be non-perturbative due to the highly non-linear nature of the Einstein-Hilbert action. Since the Wheeler-DeWitt equation cannot be solved exactly for the case at hand, we cannot give a rigorous mathematical proof that our results are totally safe. However, asymptotic safety in quantum gravity was conjectured almost thirty years ago [18] and seems to be further supported now [19] in the effective action approach of quantum field theory. We thus believe that our results can be considered as also supporting this general idea from a totally different perspective.

We shall use units with $c = \hbar = 1$.

2. Scalar field evolution at very early times

In Ref. [16], the evolution of one mode $\phi = \phi^{(k)}$ of a scalar field minimally coupled to gravity was studied by means of the Born-Oppenheimer reduction of the Wheeler-deWitt equation in the minisuperspace of a and ϕ [13]. We shall not repeat the derivation here, but just recall that using this procedure, one obtains the semiclassical Einstein-Hamilton-Jacobi equation for $a = a(\tau)$ and a perturbed Schrödinger equation for $|\chi\rangle$, the quantum state of ϕ . Both equations contain terms proportional to ℓ_P^2 , which can be viewed as representing quantum gravitational perturbations, but we shall primarily consider here their effects in the matter equation.

The case of interest to us is thus represented by the equation for $|\chi\rangle$ in the

de Sitter space-time,

$$a = a_0 e^{\mathcal{H}\tau} . \quad (2.1)$$

where \mathcal{H} is the Hubble constant. We shall also restrict our analysis to scalar field modes with wavelength much shorter than the Hubble length,

$$a/k \ll \mathcal{H}^{-1} , \quad (2.2)$$

so that the spatial curvature does not affect the scalar field dynamics and can be neglected. The relevant Schrödinger equation can then be written as (see equation (18) in Ref. [16])

$$\left(1 - \frac{3i\ell_{\text{P}}^2}{2a^3\mathcal{H}}\right) (i\partial_\tau - \hat{H}) |\chi\rangle = \frac{\ell_{\text{P}}^2}{2a^3\mathcal{H}} \Delta\hat{O} |\chi\rangle , \quad (2.3)$$

where

$$\hat{H} = \frac{1}{2} \left(\frac{\hat{\pi}^2}{a^3} + a k^2 \hat{\phi}^2 \right) \quad (2.4)$$

is the “unperturbed” Hamiltonian for the quantum state of the mode k ,

$$\hat{O} = \frac{1}{\mathcal{H}} \partial_\tau^2 + \frac{2i}{\mathcal{H}} \langle \hat{H} \rangle \partial_\tau + 3i \hat{H} , \quad (2.5)$$

and $\Delta\hat{X} \equiv \hat{X} - \langle \hat{X} \rangle$ for any operator \hat{X} , with the scalar product between matter states given by the Schrödinger product at fixed a ,

$$\langle \chi | \xi \rangle = \int \chi^*(\phi; a) \xi(\phi; a) d\phi . \quad (2.6)$$

At zero order in ℓ_{P} , the above equation describes the usual quantum evolution in the cosmic time τ of a scalar field in the de Sitter space-time, including particle production induced by the expanding background on the initial vacuum. Terms of order ℓ_{P}^2 therefore represent (leading-order) effects of the quantum gravitational fluctuations on the evolution of the scalar field. By contracting (2.3) with $\langle \chi |$, we obtain

$$i \langle \partial_\tau \rangle = \langle \hat{H} \rangle . \quad (2.7)$$

Note that this equation alone does not imply that the solutions to equation (2.3) are eigenstates of the Hamiltonian \hat{H} . In fact, (2.7) follows from $\langle \Delta\hat{O} \rangle = 0$ which means that $\Delta\hat{O}$ maps each state into orthogonal states.

To make contact with observation, it is possible to define an effective Hamiltonian which takes order ℓ_{P}^2 effects into account and consequently derive a modified dispersion relation and the CMB power spectrum [16]. In Ref. [16], all of these quantities were obtained from (2.3) by investigating the late-time evolution of the system. This was termed regime II in that paper, its precise definition being that the cosmic scale factor be so large that $a^3\mathcal{H} \gg \ell_{\text{P}}^2$. It was also stated that, for earlier times such that $a^3\mathcal{H} \ll \ell_{\text{P}}^2$ (regime I), the effect of gravitational fluctuations was expected to be negligible. To support this assertion, it was shown that the ratio between the perturbations and the Hamiltonian vanishes in regime I. However, perturbative relations, strictly valid only in regime II, were used in that argument, which might appear questionable, since one naïvely expects that the strength of quantum gravitational fluctuations grow indefinitely for increasing mode energy (*i.e.*, going backward in time). It is therefore necessary to further investigate the regime I.

2.1. Asymptotic states

We start from (2.3), which is valid in any regime and, for convenience, we rewrite as

$$(i \partial_\tau - \hat{H}) |\chi\rangle = \frac{y}{1 - 3iy} \Delta \hat{O} |\chi\rangle, \quad (2.8)$$

where

$$y \equiv \frac{\ell_P^2}{2a^3 \mathcal{H}}, \quad (2.9)$$

and the very early regime I is then defined as $y \gg 1$ (or $-\tau \gg \mathcal{H}^{-1}$).

It is important to remark that the limit $y \gg 1$ is compatible with the semiclassical approximation from which (2.3) follows. In fact, from $a/\ell_P \sim k$ and $y \gg 1$, one obtains the compatibility condition $a\mathcal{H} \ll k^{-2}$. Further, upon requiring that the mode k initially lies well inside the Hubble radius (that is, condition (2.2)), one finally obtains

$$a\mathcal{H} \ll k \ll 1/\sqrt{a\mathcal{H}}, \quad (2.10)$$

which can be satisfied when $a\mathcal{H} \ll 1$, a condition which is obviously compatible with the spatially flat de Sitter space-time (2.1) at very early times. On the other hand, if the geometry of the early Universe was exactly that of a spatially closed de Sitter [9], $a\mathcal{H} \geq 1$ and no mode k exists which satisfies (2.10). In this case, however, $y \lesssim \ell_P^2 \mathcal{H}^2 \ll 1$ and scalar field modes are generated in the perturbative regime II in which quantum gravitational fluctuations were already shown to be small [16].

We can now trade $a = a(\tau)$ and, given (2.1), the time τ for y , so that

$$\partial_\tau = -3\mathcal{H}y \partial_y, \quad (2.11)$$

and then expand all the quantities in equation (2.8) in powers of y . For example, the operator representing gravitational fluctuations becomes

$$\hat{O} = 3\mathcal{H}y \left[3\partial_y + 3y\partial_y^2 - 2iy \frac{\langle \hat{\pi}^2 \rangle}{\ell_P^2} \partial_y + i \frac{\hat{\pi}^2}{\ell_P^2} \right], \quad (2.12)$$

and the asymptotic form of the scalar field Hamiltonian is given by

$$\hat{H} = \mathcal{H}y \left[\frac{\hat{\pi}^2}{\ell_P^2} + \frac{k^2}{2} \left(\frac{\ell_P^2}{2\mathcal{H}^4 y^4} \right)^{1/3} \hat{\phi}^2 \right] \simeq \mathcal{H}y \frac{\hat{\pi}^2}{\ell_P^2}, \quad (2.13)$$

in which we neglected the potential term since it is proportional to $y^{-1/3}$. After some trivial simplifications, equation (2.8) takes on the asymptotic (and dimensionless) form

$$\left(i \partial_y + \frac{\hat{\pi}^2}{3\ell_P^2} \right) |\chi\rangle \simeq \Delta \left[\frac{\hat{\pi}^2}{3\ell_P^2} - 2 \frac{\langle \hat{\pi}^2 \rangle}{3\ell_P^2} y \partial_y - i(1 + y \partial_y) \partial_y \right] |\chi\rangle, \quad (2.14)$$

which can be easily solved by writing

$$|\chi\rangle = e^{i\varphi(y)} |\kappa\rangle, \quad (2.15)$$

where $|\kappa\rangle$ are orthogonal eigenstates of the scalar field momentum,

$$\frac{\hat{\pi}}{\sqrt{3}\ell_P} |\kappa\rangle = \kappa |\kappa\rangle, \quad (2.16)$$

and do not depend on y . The right end side of equation (2.14) therefore vanishes on such states and one finds

$$\varphi = \kappa^2 y, \quad (2.17)$$

so that the asymptotic ($-\tau = -\tau_{\text{asy}} \gg \mathcal{H}^{-1}$) un-normalised quantum states of the scalar field mode k can be finally written as

$$|\chi_{\text{asy}}\rangle \sim \exp\left(i \frac{\ell_{\text{P}}^2 \kappa^2}{2 a^3 \mathcal{H}}\right) |\kappa\rangle = \exp\left(-i \int^{\tau_{\text{asy}}} \frac{3 \ell_{\text{P}}^2 \kappa^2}{2 a^3} dt\right) \sqrt{\frac{\sqrt{3} \ell_{\text{P}}}{2 \pi}} e^{i \sqrt{3} \kappa \ell_{\text{P}} \phi}, \quad (2.18)$$

where the factor of $\sqrt{\ell_{\text{P}}}$ was included to make the scalar product (2.6) dimensionless, since ϕ is the inverse of a length. Finally, we note that these solutions (2.18) to the matter equation (2.3) in the asymptotic regime I also satisfy the unperturbed Schrödinger equation

$$i \partial_{\tau} |\chi_{\text{asy}}\rangle = \hat{H}_{\text{asy}} |\chi_{\text{asy}}\rangle, \quad (2.19)$$

with the free particle Hamiltonian

$$\hat{H}_{\text{asy}} = \frac{\hat{\pi}^2}{2 a^3} \simeq \frac{\hat{\pi}^2}{2 \ell_{\text{P}}^3 k^3}. \quad (2.20)$$

Consequently, for very early times (when $a \sim k \ell_{\text{P}}$), the influence of gravitational fluctuations does not grow indefinitely, but vanishes asymptotically.

Technically speaking, the above plane-waves (2.18) are not normalisable for ϕ and $\kappa \in \mathbb{R}$ and their physical interpretation remains somewhat unclear so far, also because of their dependence on the parameter κ . Further, it is easily seen that they do not solve equation (2.3) for finite values of y since, in this case, the potential in \hat{H} is not negligible and the Hamiltonian changes (approximately) into that of a harmonic oscillator (free scalar field) with time-dependent frequency (and mass) which admits (approximate) normalizable (discrete) eigenstates. Thus, as soon as the potential is no more negligible, the Hamiltonian spectrum changes and particle production begins (moreover, $\Delta \hat{O}|\chi\rangle \neq 0$, and transitions also occur among invariant eigenstates ‡). The only way one may accommodate for the asymptotic states (2.18) in the theory is therefore by considering normalised wave-packets as initial conditions for the subsequent evolution of the scalar field states §. We shall however see that one can still draw significant conclusions without dealing with such technicalities in a rigorous manner.

2.2. Early time evolution

The subsequent evolution, for larger values of y , can be studied by expressing \hat{H} of equation (2.3) in terms of the invariant creation and annihilation operators \hat{b}^\dagger and \hat{b} [20, 15, 16] and expand it in powers of $x \equiv a \mathcal{H}/k \ll 1$ (see the condition (2.10)),

$$\hat{H} = \frac{k}{a} \left[\hat{b}^\dagger \hat{b} + \frac{1}{2} + \frac{i}{2} x \left(\hat{b}^2 - (\hat{b}^\dagger)^2 \right) \right] + O(x^2). \quad (2.21)$$

Upon neglecting the term of order x , the invariant operators determine a basis $\{|n\rangle\}$ of orthonormal solutions to

$$i \partial_{\tau} |\chi_s\rangle = \hat{H} |\chi_s\rangle, \quad (2.22)$$

such that (again, for $\tau \sim \tau_{\text{asy}}$)

$$\hat{H} |n\rangle \simeq \frac{k}{a} \left(n + \frac{1}{2} \right) |n\rangle \equiv \omega_n |n\rangle, \quad (2.23)$$

‡ For more details about invariant operators and states, see Refs. [20, 15, 16].

§ This interpretation partly overcomes one of the two criticisms to the approach of Ref. [6].

for which quantum gravitational fluctuations are still negligible

$$\Delta \hat{O}|n\rangle = \mathcal{O}(x^2) . \quad (2.24)$$

In this approximation, the complete (normalised) solution to our problem in regime I will then be given by

$$|\chi_s\rangle \simeq \mathcal{N}^{1/2} \sum_n e^{-i\omega_n(\tau-\tau_{\text{asy}})} |n\rangle \langle n | \chi_{\text{asy}} \rangle , \quad (2.25)$$

where $\tau \gtrsim \tau_{\text{asy}} \rightarrow -\infty$ and $\mathcal{N} = \mathcal{N}(\kappa)$ is a normalisation factor. This form, which involves the usual states $|n\rangle$ with fixed number of quanta, will now allow us to give a more transparent interpretation of the initial states $|\chi_{\text{asy}}\rangle$ in terms of physically meaningful quantities.

We shall first obtain a relation between the asymptotic eigenvalue κ and quantities that appear in the spectrum of the Hamiltonian (2.21). For this purpose, it appears natural to equate the eigenvalue of the asymptotic Hamiltonian (2.20) on the asymptotic states $|\kappa\rangle$ with the expectation value of \hat{H} on the states in (2.25) for $\tau \rightarrow \tau_{\text{asy}}$. In particular,

$$\begin{aligned} \langle n | \chi_{\text{asy}} \rangle &\simeq \left(\frac{3}{\pi^3} \right)^{1/4} \sqrt{\frac{\sqrt{k} a \ell_P}{2^{n+1} n!}} \int_{-\infty}^{+\infty} d\phi e^{i\sqrt{3}\kappa \ell_P \phi} e^{-\frac{\kappa}{2} a^2 \phi^2} H_n(\sqrt{k} a \phi) \\ &= \frac{i^n}{\sqrt{2^{n+1} n!}} \left(\frac{3}{\pi^3} \right)^{1/4} \sqrt{\frac{\ell_P}{\sqrt{k} a}} e^{-3\ell_P^2 \kappa^2 / 2 k a^2} H_n\left(\sqrt{3} \frac{\ell_P \kappa}{\sqrt{k} a}\right) , \end{aligned} \quad (2.26)$$

since the weighted Hermite polynomials are eigenfunctions of the Fourier transform. This expression must be evaluated in the asymptotic regime by assuming $a \sim k \ell_P$, thus yielding

$$\langle n | \chi_{\text{asy}} \rangle \simeq i^n \left(\frac{3}{\pi^3} \right)^{1/4} \frac{e^{-3\kappa^2 / 2 k^3}}{\sqrt{2^{n+1} n! k^{3/2}}} H_n\left(\sqrt{3} \kappa k^{-3/2}\right) . \quad (2.27)$$

Since each state $|n\rangle$ of the mode k contributes an amount of energy equal to $\omega_n^{(k)} \simeq n k / a \simeq n / \ell_P$, the energy (in units of the Hubble mass \mathcal{H}) stored in $|\chi_{\text{asy}}^{(k)}\rangle$ for k and κ fixed is given by

$$\frac{E^{(k)}}{\mathcal{H}} \simeq \frac{\mathcal{N}^{(k)}}{2 \ell_P \mathcal{H}} \sqrt{\frac{3}{\pi^3}} \frac{e^{-3\kappa^2 / k^3}}{k^{3/2}} \sum_{n=1}^{\infty} n R_n^{(k)}(\kappa) , \quad (2.28)$$

where

$$R_n^{(k)}(\kappa) = \left| \frac{\langle n | \chi_{\text{asy}} \rangle}{\langle 0 | \chi_{\text{asy}} \rangle} \right|^2 = \frac{H_n^2(\sqrt{3} \kappa k^{-3/2})}{2^n n!} . \quad (2.29)$$

We finally equate $E^{(k)}$ to the asymptotic energy

$$E_{\text{asy}}^{(k)} = \frac{3 \ell_P^2 \kappa^2}{2 a^3} \simeq \frac{3 \kappa^2}{2 \ell_P k^3} , \quad (2.30)$$

so that equation (2.28) determines the normalisation $\mathcal{N}^{(k)}$ as a function of k and $E_{\text{asy}}^{(k)}$,

$$\mathcal{N}^{(k)} \simeq 2 \sqrt{\frac{\pi^3}{3}} k^{3/2} e^{2 \ell_P E_{\text{asy}}^{(k)}} \ell_P E_{\text{asy}}^{(k)} \left[\sum_{n=1}^{\infty} n R_n^{(k)} \right]^{-1} , \quad (2.31)$$

|| We are subtracting the zero-mode energy.

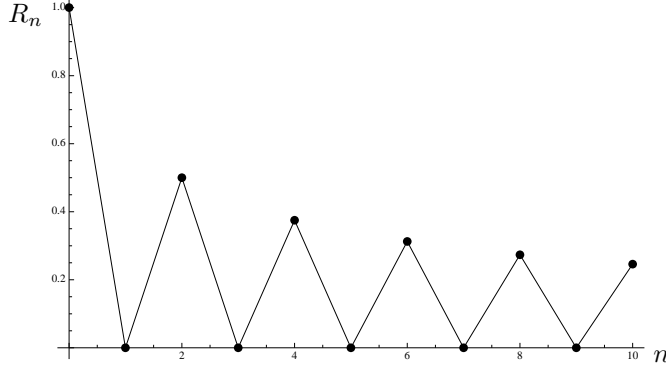


Figure 1. Distribution in equation (2.35) for $n = 0, \dots, 10$. Only even excitations appear in the asymptotic states.

where

$$R_n^{(k)} = \frac{H_n^2 \left(\sqrt{2 \ell_P E_{\text{asy}}^{(k)}} \right)}{2^n n!} . \quad (2.32)$$

All the expressions are now functions of the wave mode k and the asymptotic energy $E_{\text{asy}}^{(k)}$ which remain as free parameters.

We recall that k is constrained by equation (2.10) and the total matter energy must be small with respect to \mathcal{H} , otherwise the scale factor would not evolve *a la* de Sitter. Assuming the range of available k extends over at least a few orders of magnitude, the total energy

$$E_{\text{asy}} = \int_{a \mathcal{H}}^{(a \mathcal{H})^{-1/2}} E_{\text{asy}}^{(k)} dk , \quad (2.33)$$

can be small only if ¶

$$E_{\text{asy}}^{(k)} \ll \frac{a^3 \mathcal{H}}{2 \ell_P^2} \sim k^3 \ell_P^2 \mathcal{H}^2 , \quad (2.34)$$

for all the allowed k . The relative probability of finding n excitations of each mode k is then approximately given by

$$R_n^{(k)} \simeq \frac{H_n^2(0)}{2^n n!} = \frac{2^n \pi}{n! \Gamma^2((1-n)/2)} . \quad (2.35)$$

which is shown in Fig. 1.

Technically, since $|\chi_{\text{asy}}\rangle$ is not normalisable, the series in equation (2.28) would diverge and $\mathcal{N}^{(k)}$ would vanish unless the asymptotic states were regularised. As we mentioned, both problems could be in fact cured by using wave-packets instead of the simple plane-waves (2.18). However, any such regularization would only affect the distributions $R_n^{(k)}$ at given k for n large, making them decrease faster for increasing n than the case shown in equation (2.35). We can therefore conclude that the states $|\chi_{\text{asy}}^{(k)}\rangle$ are in general peaked on the vacuum $|n=0\rangle$ and contain very small amounts of (even) excitations for all the modes k .

¶ See equation (2.39) below and Ref. [21].

2.3. Matter fluctuations in the gravitational equation

The asymptotic matter states we found are consistent solutions only if we can show that their contribution to the evolution of the scale factor of the Universe is small compared to the driving cosmological constant $3\mathcal{H}^2$. This is not just tantamount to the condition (2.34) above, but also requires that their quantum gravitational fluctuations in the Einstein-Hamilton-Jacobi equation for $a = a(\tau)$ remain small [13].

From Ref. [15], we know that the equation that determines the evolution of the wave-function $\psi = \psi(a)$ for the cosmic scale factor is given by

$$\left(\frac{\ell_{\text{P}}^2}{2a} \partial_a^2 + \frac{a^3 \mathcal{H}^2}{2\ell_{\text{P}}^2} + \sum_k \langle \chi_k | \hat{H} | \chi_k \rangle \right) \psi = -\frac{\ell_{\text{P}}^2}{2a} \sum_k \langle \chi_k | \partial_a^2 | \chi_k \rangle \psi, \quad (2.36)$$

where the right hand side represents the aforementioned fluctuations, the sum in k formally represents the integral in equation (2.33) and expectation values are taken on asymptotic states in the Heisenberg representation [16],

$$|\chi_k\rangle = \exp\left(i \int^\tau \langle \chi_s | \hat{H} | \chi_s \rangle dt\right) |\chi_s\rangle \simeq \exp\left(i \int^\tau E_{\text{asy}}^{(k)} dt\right) |\chi_{\text{asy}}\rangle. \quad (2.37)$$

From equations (2.18) and (2.30), one therefore obtains

$$\partial_a^2 |\chi_k\rangle \simeq i \partial_a \left[\left(\frac{E_{\text{asy}}^{(k)}}{a\mathcal{H}} - \frac{3\ell_{\text{P}}^2 \kappa^2}{2a^4 \mathcal{H}} \right) |\chi_k\rangle \right] \simeq 0. \quad (2.38)$$

It then follows that the gravitational equation reduces to

$$\left(\frac{\ell_{\text{P}}^2}{2a} \partial_a^2 + \frac{a^3 \mathcal{H}^2}{2\ell_{\text{P}}^2} + E_{\text{asy}} \right) \psi = 0, \quad (2.39)$$

which yields the WKB solution corresponding to the de Sitter evolution (2.1) provided the condition (2.34) on the total matter energy we discussed before holds⁺.

2.4. Intermediate regime

The results presented this far are quite interesting in themselves but do not yet fully validate the findings of Ref. [16]. In fact, from Ref. [16], we already know that the effects on matter states produced by gravitational fluctuations in regime II ($y \ll 1$) are very small (of order ℓ_{P}^2), so we need to make sure that in between regimes I and II the fluctuations never become overwhelming with respect to the left hand side in equation (2.8).

We do not attempt at solving analytically equation (2.8) for arbitrary values of y , but we just note that there is no singularity in the strength of the fluctuations at the transition between regimes I and II. Fig. 2 shows the real and imaginary parts of the coefficient that “couples” with the gravitational fluctuations in (2.8),

$$\gamma(y) \equiv \frac{y}{1 - 3iy}. \quad (2.40)$$

Looking at the graphs, it is evident that not only there is no singularity, but the real and imaginary parts of γ never grow larger than a few decimals. Therefore, the gravitational perturbations never become a dominant factor in the dynamics of the scalar field in de Sitter and the system reaches the regime II analyzed in Ref. [16] in a state that does not differ significantly from (or even is just the same as) the one it was in during regime I. This clearly supports the results presented in Ref. [16].

⁺ A more complete analysis of the coupled gravity-matter dynamics is being performed [21].

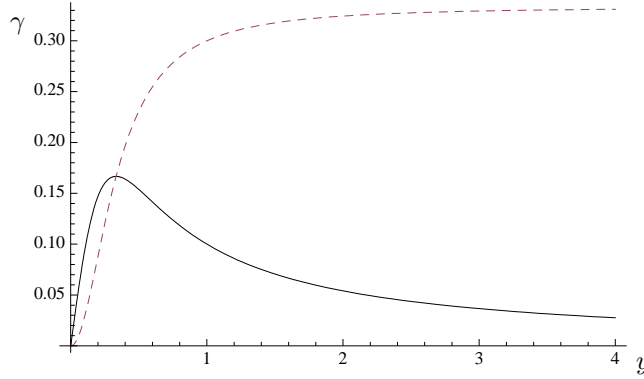


Figure 2. Real (solid line) and imaginary (dashed line) parts of $\gamma(y)$. Real part tends to zero for y small (regime II) and large (regime I) and has a maximum of $1/6$ at $y = 1/3$. Imaginary part vanishes at $y = 0$ and tends to $1/3$ for large y .

3. Summary and conclusions

We have investigated the influence of quantum gravitational fluctuations on the evolution of a minimally coupled scalar field mode k in a spatially flat de Sitter background for values of the scale factor near the limit of validity of the semiclassical approximation, that is for $a \gtrsim k \ell_P$. We have found that, contrary to naïve intuition, the effects of such fluctuations do not grow indefinitely with increasing energy (*i.e.*, going backward in time) but they actually vanish and the scalar field becomes asymptotically free in what we called regime I. This result is further supported by the fact that quantum gravitational fluctuations have been shown to asymptotically vanish in the semiclassical equation that governs the evolution of the cosmic scale factor [13, 15, 16]. It is important to remark that these conclusions only apply to those wave-numbers k that satisfy the inequalities (2.10). Of course, this condition is not really restrictive in the flat de Sitter space we have considered here, since a can be as small as possible. However, (on general quantum mechanical considerations [11]) one expects that the Universe was born into the semiclassical regime with a non-zero initial scale factor and several approaches to quantum cosmology further support a spatially closed initial geometry (see, *e.g.*, Refs. [9, 10]). The latter situation was actually taken into account in deriving equation (2.10), which ensures the scalar field modes of relevance have a wavelength much shorter than the Hubble length and are not affected by the spatial curvature at very early times.

Of course, the possibility remains that $a\mathcal{H} \gtrsim 1$ initially, which would rule out all values of k with the asymptotic behaviour found in Section 2.1. This occurs, for example, if the initial state of the Universe is exactly the closed de Sitter space-time of Ref. [9]. For such large initial values of a , the regime I never happens and the Universe is actually born in regime II, in which quantum gravitational fluctuations are always very small (of order $\mathcal{H}^2 \ell_P^2$; see Ref. [16]). Since we have also shown that between the early-time limit (if it exists) and the perturbative late-time phase, both the real and the imaginary part of the coupling to gravitational fluctuations (2.40) always remain small, we can conclude that the gravitational perturbations never become a dominant factor for the dynamics of the scalar field in a de Sitter Universe. This argument validates and supports the result of Ref. [16] where the effective dispersion relation

and CMB spectrum were computed.

We wish to conclude by adding that, even if regime I never occurred in our actual Universe, the fact that a matter-gravity system admits the kind of asymptotic freedom we have found can be of theoretical interest for our understanding of gravitation. In fact, although our approach is entirely different, we recalled in the Introduction that a form of asymptotic freedom for pure gravity was conjectured in Ref. [18] and later supported, *e.g.*, in Refs. [19].

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